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A Repulsive Force from a Modification of General Relativity

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Abstract Assuming a modification of general relativity which implies a direct coupling between the Ricci curvature scalar and the matter Lagrangian, a non-geodesic motion of test particles is obtained, which, performing a repulsive force, could, in principle, be connected with Dark Matter and Pioneer anomaly problems.

Keywords Repulsive force · Non-geodesic motion

1 Introduction

The accelerated expansion of the Universe, which is today observed, shows that cosmological dynamic is dominated by the so called Dark Energy which gives a large negative pressure. This is the standard picture, in which such new ingredient is considered as a source of the *rhs* of the field equations. It should be some form of un-clustered non-zero vacuum energy which, together with the clustered Dark Matter, drives the global dynamics. This is the so called "concordance model" which gives, in agreement with the CMBR, LSS and SNeIa data, a good trapestry of the today observed Universe, but presents several shortcomings as the well known "coincidence" and "cosmological constant" problems [1]. An alternative approach is changing the *lhs* of the field equations, seeing if observed cosmic dynamics can be achieved extending general relativity [2–4]. In this different context, it is not required to find out candidates for Dark Energy and Dark Matter, that, till now, have not been found, but only the "observed" ingredients, which are curvature and baryonic matter, have to be taken into account. Considering this point of view, one can think that gravity is not scale-invariant [5] and a room for alternative theories is present [6–8]. In principle, the most popular Dark Energy and Dark Matter models can be achieved considering $f(R)$

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theories of gravity [5, 9, 17], where R is the Ricci curvature scalar.

In this picture even the sensitive detectors for gravitational waves, like bars and interferometers (i.e. those which are currently in operation and the ones which are in a phase of planning and proposal stages) [10, 11], could, in principle, be important to confirm or ruling out the physical consistency of general relativity or of any other theory of gravitation. This is because, in the context of Extended Theories of Gravity, some differences between general relativity and the others theories can be pointed out starting by the linearized theory of gravity [12–16].

In this paper a simple modification of general relativity will be assumed, considering an explicit coupling between the Ricci scalar and the matter Lagrangian. The result is a non-geodesic motion of test particles which, performing a repulsive force, could, in principle, be connected with Dark Matter and Pioneer anomaly problems.

Let us consider the action

$$S = \int d^4x\sqrt{-g}(R + R\mathcal{L}_m + \mathcal{L}_m). \tag{1}$$

The modification in the action only includes a coupling between the Ricci scalar and the matter Lagrangian with respect the well known canonical one of general relativity (the Einstein-Hilbert action [18, 19]) which is

$$S = \int d^4x\sqrt{-g}R + \mathcal{L}_m. \tag{2}$$

2 The Field Equations

Let us consider the variational principle (note that in this paper we work with $4\pi G = 1$, $c = 1$ and $\hbar = 1$) [19, 24]

$$\delta \int d^4x\sqrt{-g}(R + R\mathcal{L}_m + \mathcal{L}_m) = 0 \tag{3}$$

in a local Lorentz frame.

One gets:

$$\begin{aligned}
&\delta \int d^4x\sqrt{-g}(R + R\mathcal{L}_m + \mathcal{L}_m) \\
&\quad = \int d^4x[\delta\sqrt{-g}(R + R\mathcal{L}_m + \mathcal{L}_m) + \sqrt{-g}\delta(R + R\mathcal{L}_m + \mathcal{L}_m)] \\
&\quad = \int d^4x\left[\sqrt{-g}(1 + \mathcal{L}_m)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + R\mathcal{L}_m + \mathcal{L}_m)\right]\delta g^{\mu\nu} \\
&\qquad + \int d^4x\sqrt{-g}(1 + \mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu}.
\end{aligned} \tag{4}$$

Recalling the relation between the Christoffel coefficients and the Ricci tensor [19, 24] one can write

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\partial_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - g^{\mu\alpha}\partial_\alpha(\delta\Gamma^\nu_{\mu\nu}) \equiv \partial_\alpha X^\alpha, \tag{5}$$

where

$$X^{\alpha} \equiv g^{\mu\nu}(\delta\Gamma^{\alpha}_{\mu\nu}) - g^{\mu\alpha}(\delta\Gamma^{\nu}_{\mu\nu}). \tag{6}$$

In this way, the second integral in (4) can be computed as

$$\begin{aligned}
&\int d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu} \\
&\quad= \int d^4x\sqrt{-g}(1+\mathcal{L}_m)\partial_\alpha X^{\alpha} \\
&\quad= \int d^4x\,\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)X^{\alpha}] - \int d^4x\,\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]X^{\alpha}.
\end{aligned} \tag{7}$$

Assuming that fields are equal to zero at infinity, one gets

$$d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu} = -\int d^4x\,\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]X^{\alpha}. \tag{8}$$

Now, let us compute X^{α}. Recalling that in a local Lorentz frame it is

$$\nabla_\beta g_{\mu\nu} = \partial_\beta g_{\mu\nu} = 0 \tag{9}$$

and, using the well known definitions of the Christofell coefficients [19, 24], it is

$$\begin{aligned}
\delta\Gamma^{\alpha}_{\mu\nu} &= \delta\left[\frac{1}{2}g^{\beta\alpha}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})\right] \\
&= \frac{1}{2}g^{\beta\alpha}(\partial_\mu \delta g_{\beta\nu} + \partial_\nu \delta g_{\mu\beta} - \partial_\beta \delta g_{\mu\nu}).
\end{aligned} \tag{10}$$

In the same way it is

$$\delta\Gamma^{\nu}_{\mu\nu} = \frac{1}{2}g^{\nu\beta}\partial_\mu(\delta g_{\nu\beta}). \tag{11}$$

From (10) and (11) one gets

$$g^{\mu\nu}(\delta\Gamma^{\alpha}_{\mu\nu}) = \frac{1}{2}\partial^{\alpha}(g_{\mu\nu}\delta g^{\mu\nu}) - \partial^{\mu}(g_{\beta\mu}\delta g^{\nu\beta}) \tag{12}$$

and

$$g^{\mu\alpha}(\delta\Gamma^{\nu}_{\mu\nu}) = -\frac{1}{2}\partial^{\alpha}(g_{\nu\beta}\delta g^{\nu\beta}). \tag{13}$$

Then, substituting in (6), it is

$$X^{\alpha} = \partial^{\alpha}(g_{\mu\nu}\delta g^{\mu\nu}) - \partial^{\mu}(g_{\mu\nu}\delta g^{\alpha\nu}). \tag{14}$$

With this equation, (8) becomes

$$\begin{aligned}
&\int d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu} \\
&\quad= \int d^4x\,\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)][\partial^{\mu}(g_{\mu\nu}\delta g^{\alpha\nu}) - \partial^{\alpha}(g_{\mu\nu}\delta g^{\mu\nu})],
\end{aligned} \tag{15}$$

which also gives

$$\int d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu}$$
$$=\int d^4x\{g_{\mu\nu}\partial^\alpha\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]\delta g^{\mu\nu}\}-\int d^4x\{g_{\mu\nu}\partial^\mu\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]\delta g^{\alpha\nu}\}. \tag{16}$$

Putting this equation in the variation (4) one gets

$$\delta\int d^4x\sqrt{-g}(R+R\mathcal{L}_m+\mathcal{L}_m)$$
$$=\int d^4x\left[\sqrt{-g}(1+\mathcal{L}_m)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}(R+R\mathcal{L}_m+\mathcal{L}_m)\right]\delta g^{\mu\nu}$$
$$+\int d^4x\{g_{\mu\nu}\partial^\alpha\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]-g_{\alpha\nu}\partial^\mu\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]\}\delta g^{\mu\nu}\}$$
$$+\int d^4x(1+R)\delta(\sqrt{-g}\mathcal{L}_m). \tag{17}$$

The above variation is equal to zero for

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-\mathcal{L}_mR_{\mu\nu}+(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box)\mathcal{L}_m+\frac{(1+R)}{2}T^{(m)}_{\mu\nu}, \tag{18}$$

which are the modified Einstein field equations. In these equations the standard stress-energy tensor [18, 19]

$$T^{(m)}_{\mu\nu}\equiv\frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{19}$$

has been introduced.

3 A Non-geodesic Force

The field equations (18) can be put in the well known Einsteinian form

$$G_{\mu\nu}=8\pi T^{(tot)}_{\mu\nu}, \tag{20}$$

if one writes down explicitly the Einstein tensor and introduces a "total" stress-energy tensor

$$T^{(tot)}_{\mu\nu}\equiv\frac{1}{(1+\mathcal{L}_m)}\left[(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box)\mathcal{L}_m+\frac{(1+R)}{2}T^{(m)}_{\mu\nu}-\frac{R\mathcal{L}_m}{2}g_{\mu\nu}\right], \tag{21}$$

in which a *curvature* contribute is added and mixed to the *material* one. This is because the high order terms contribute, like sources, to the field equations and can be considered like *effective fields* (see [25] for details).

In this way, putting the condition of energy conservation

$$\nabla^\mu G_{\mu\nu}=0 \tag{22}$$

in (20) and (21), one obtains

$$\nabla^{\mu} T^{(m)}_{\mu\nu} = \frac{1}{R+1}(g_{\mu\nu}\mathcal{L}_m - T^{(m)}_{\mu\nu}) \nabla^{\mu} R. \tag{23}$$

With the goal of testing the motion of test particles in the model, one can introduce the well known stress-energy tensor of a perfect fluid [18, 19]

$$T^{(m)}_{\mu\nu} \equiv (\epsilon + p)u_{\mu}u_{\nu} - p g_{\mu\nu}. \tag{24}$$

Because two astrophysical examples will be considered (the first in the galaxy and the second in the Solar System), this simplest version of a stress-energy tensor for the matter, which concerns inchoerent matter, can be used in a good approximation [18, 19, 25]. In (24) ϵ is the proper energy density, p the pressure and u_{μ} the fourth-velocity of the particles.

Now, following [9, 20], one defines the *projector operator*

$$P_{\mu\alpha} \equiv g_{\mu\alpha} - u_{\mu}u_{\alpha}, \tag{25}$$

the contraction $g^{\alpha\beta}P_{\mu\beta}$ can be applied to (23), obtaining

$$\frac{d^2x^{\alpha}}{ds^2} + \tilde{\Gamma}^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = F^{\alpha}. \tag{26}$$

The presence of the extra force

$$F^{\alpha} \equiv (\epsilon + p)^{-1} P^{\alpha\nu}\left[\left(\frac{1}{R+1}\right)(\mathcal{L}_m + p) \nabla_{\nu} R + \nabla_{\nu} p\right] \tag{27}$$

shows that the motion of test particles is non-geodesic. It is also simple to see that

$$F^{\alpha}\frac{dx_{\alpha}}{ds} = 0, \tag{28}$$

i.e. the extra force is orthogonal to the four-velocity of test masses.

Taking the Newtonian limit in three dimensions of (26) one obtains

$$\vec{a}_{tot} = \vec{a}_n + \vec{a}_{ng}, \tag{29}$$

where the total acceleration $\vec{a}_{tot}$ is given by the ordinary Newtonian acceleration $\vec{a}_n$ plus the repulsive acceleration $\vec{a}_{ng}$ which is due to the extra force (non-geodesic).

Using (29) and a bit of three-dimensional geometry the Newtonian acceleration $\vec{a}_n$ can be written as

$$\vec{a}_n = \frac{1}{2}(a^2_{tot} - a^2_n - a^2_{ng})\frac{\vec{a}_{tot}}{a_{ng}a_{tot}}. \tag{30}$$

In the limit in which $\vec{a}_{ng}$ dominates (i.e. $a_n \ll a_{tot}$) it is

$$a_n \simeq \frac{a_{tot}\vec{a}_{tot}}{2a_{ng}}\left(1 - \frac{a^2_{ng}}{a^2_{tot}}\right). \tag{31}$$

Defining [9, 20–22]

$$a_e^{-1} \equiv \frac{1}{2a_{ng}}\left(1 - \frac{a^2_{ng}}{a^2_{tot}}\right), \tag{32}$$

(31) becomes

$$\vec{a}_n \simeq \frac{a_{tot}}{a_e}\vec{a}_{tot}. \tag{33}$$

Recalling that the standard Newtonian acceleration is

$$\vec{a}_n = \frac{M}{r^2}\widehat{u}_r, \tag{34}$$

the total acceleration results

$$\vec{a}_{tot} = \frac{(a_e M)^{\frac{1}{2}}}{r}\widehat{u}_r = \frac{v_r^2}{r}\widehat{u}_r, \tag{35}$$

where

$$v_r = (a_e M)^{\frac{1}{4}} \tag{36}$$

is the rotation velocity of a test mass under the influence of the non-geodesic force.

Because of the environment nature of the non-geodesic force, one recalls motivations that arise from phenomenology to identificate it. In a galactic context it is natural to identify a_e with $a_0 \simeq 10^{-10}$ m/s^2, which is the acceleration of Milgrom used in the theoretical context of Modified Newtonian Dynamics to achieve Dark Matter into galaxies [9, 21, 22].

From another point of view, in the Solar System, if the anomaly in Pioneer acceleration is not generated by systematic effects, but a real effect is present [9, 21–23], one can in principle put

$$a_e = a_{Pi} \simeq 8.5 \times 10^{-10}\ \mathrm{m/s^2}. \tag{37}$$

Thus, the introduced approach allows for a unified explanation of the two effects.

Conclusions

In this paper a modification of general relativity which implies a direct coupling between the Ricci curvature scalar and the matter Lagrangian has been considered. A non-geodesic motion of test particles has been obtained, which, performing a repulsive force, could, in principle, be connected with Dark Matter and Pioneer anomaly problems.

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